From Cognitive Mappings to Sheaves

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Abstract: The grand Marxist science of history is today relegated to a more modest call to render visible the rapidly growing economic complexity around us. This text argues that a combination of data science and topology might provide tools to track this complexity and give us a new, scientific reading of Marx's theory. We propose a novel interpretation of value as a space, and the market as a process of "sheafification". We show how this approach provides an intuitive framework for a "data-driven" approach to the critique of political-economy.

Keywords: Marx, Jameson, Dupuy, Space, Data, Topology, Sheaf, Market

Marxism as a "cognitive mapping"

The phrase "cognitive mapping"[1] has served in recent decades as a call to work for cultural theorists and philosophers. It signifies something we lack today: a way to track the determining role of economic abstractions in everyday life. Though we are reminded constantly of what happens on the market, these events remain blurry and at a distance such that we only ever see vague patterns in them. Information becomes indecipherable in a political sense even as our access to it grows. Proportional to the deluge of information is our reliance on experts to interpret what the market wants from us. However, these interpretations inevitably fall short. It is true that one can *describe* the global economy from data which is readily available[2], and such descriptions may yield valuable predictions. However, because they are restricted to the world of commodities, these descriptions together offer only a *flattened* image of capitalism.

Marx, on the other hand, proposes that any faithful model of capitalism must explain its inherent tendency towards crisis. He also predicted that such crises would necessarily lead to a complete rewrite of the social order. Regardless of whether his prediction comes true, Marx's effort to systematically think not simply phenomena *within* capitalism (price fluctuations, growth, unemployment, etc.), but also the historical ruptures which precede and succeed it, remains compelling. By constructing a model larger than that of a single economic system, in which capitalism can be viewed as just one moment, Marx opened a hole in political-historical thought.

Today, critiques of the excess of capitalism are by no means

1 First coined by Lynch 1960 it was then borrowed by Jameson 1992, then Slavoj Zizek and others. For a good history of the term, see Toscano and Kinkle 2014

2 For example https://atlas.media.mit.edu/en/ or https://data.oecd.org/api/

exclusively Marxist[3]. As bourgeois economists attempt to grasp the systemic contradictions of wealth inequality, their explanations may begin to resemble Marx's, though always careful to not overstep ideological boundaries. Within Marxism itself there are still internal debates around the interpretation and application of Marx's arguments[4]. Thus we find a zone of indistinction between non-Marxists who sound like Marx and those Marxists who are not Marxist enough. Although this may appear as the result of an empty "academic" exercise, it is actually a necessary moment in freeing ourselves from old prejudices. Our lack of cognitive mapping constitutes the ground of an ideological struggle over the "means of interpretation" of economic facts. From the bourgeois standpoint, this struggle involves delimiting the natural order of economic relations and thereby isolating unnatural distortions of these relations. From the Marxist standpoint, it is the de-naturalization and politicization of the economy. However, new positions are appearing which are difficult to classify as either stance, and this perhaps is where new means of interpretation can be found.

In its ideological battle, Marxism has (ironically) ceded ground in terms of scientific tools. This can be understood as part of its adherence to a framework: for many Marxists, the information produced in economic activity is *noise*, and therefore the goal is to filter it out and look only for trends which confirm Marx's theory. To describe the market independent of this effort (i.e. scientifically) is counterproductive, not least because a formal economic treatment is inaccessible to a wide audience, but mainly because scientific formalism generally treats objects as ahistorical. Because of this, we hastily conclude that Marxism's "cognitive mapping" is incompatible with a scientific approach towards the economy. It is true that *only* Marxism offers an account wherein the economy is *produced* by politics and political struggles. Yet it often conflates this truth with a mastery of science itself, an attitude both dogmatic and identitarian.

The Marxist model of history, although it still aspires to be scientific, reduces its object to that of classical mechanics, wherein movement is unaffected by measurement. This leads to the following impredicative paradox: if we suppose "communism" to mean a social form constructed through acknowledging the law of value and class struggle, then this form should be at least as complex and unpredictable as the forms it replaces - otherwise, it could not *include* these previous forms. If so, how do we reconcile this requirement of complexity with Marxism's

3 Some popular accounts are Reich 2015 and Piketty 2014

4 See, for example, the debate raised by Heinrich 2013 and some notable responses from Kliman, Freeman, Potts, Gusev, and Cooney 2013, and Carchedi, Roberts 2013

teleology and voluntarism? Marx's narrative of the dictatorship of the proletariat collapses, not due to its empirical failures insofar as these can be explained by a Ptolemaic revision, but because of its recursive nature. As long as the proletariat depends on crisis to assert itself, it will also depend on the social forms which produce crisis. Any *emergent* version of communism therefore implicitly takes the market as the paradigm for self-referential complexity in social organization[5]. In short, the old Austrian-school critique of planning remains unanswered, namely, that one cannot have dynamic growth without the price system. This is attested by the fact that as soon as we envision society *without* the market, Marx's theory becomes inconsistent.

The outcome is that Marxists today do not usually spend time formulating ways of running a global economy[6]. When we advance to concrete proposals we find sectarian splits over questions such as the role of money, party, and state. This is not necessarily a critique but an observation valid for anyone who attempts to move from critiques to construction. To be fair, modern (neoclassical) economics has not even reached the conclusion that capitalism should be reshaped, preferring to turn a blind eye to the regularity and intensification of crises. Marx's analysis still holds the advantage here because it already counts overproduction, unemployment, class divisions, and ecological degradation as internal to our economy[7], whereas these are only "externalities" to bourgeois economists. The approach of this text is the same - we take capital to be a complex, inherently "contradictory" form that does not necessarily converge to a state beneficial to humans.

Given the events of the 20th century, it seems the dual goals of scientifically analyzing history and planning the economy inevitably lead to trusting in reductive models. But perhaps this is only a consequence of the scientific illiteracy that Marxism has resigned itself to? Today, the fields of computational data science and machine learning offer methods to construct models of social phenomena whose complexity threshold are much higher than we can imagine. These fields begin with a space larger than our individual perspectives, the multidimensional

5 In other words, society belongs to the class of phenomena whose laws do not allow for an immediate prediction of a future state - rather, we can only simulate *parts* of its future state.

6 Those that do so cease to appear Marxist at all. For example, "socialism with Chinese characteristics" or "capitalism with Asian values" shows how Marxist pragmatism becomes a market pragmatism without much need for Marx any longer. So we either have a clear view of historical transformation without grasping market complexity, or we have a concrete way of controlling the market via authoritarian power but no recognizable vision for transformation.

7 Recall Zizek's four horsemen: ecological destruction, apartheid, unchecked biogenetic technologies and the indetermination of intellectual property. Also see Saito 2016 for an ecological reading of Marx's notebooks.

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space of data itself. From this space we can computationally grasp otherwise invisible social phenomena. Perhaps this is not only a tool for inventively extracting surplus value, but also the means of realizing the dream of Marx, a "new kind of science" of history. What if certain datasets articulate spaces *between* the lifeworld of individuals and the totality of capitalist relations? This domain would be historical but not psychological, concrete yet amenable to formal methods, and thus a site for political intervention.

Data itself is meaningless without a set of metrics that can "project" it into some space[8]. These metrics likewise depend on a process of choosing the right questions to ask, questions which determine the kind of meaningful answers one can receive[9]. We should combine this with the fact that the value abstraction is not a natural phenomena but contingent and political. One could therefore speculatively define today's alternate forms of organization as those which can produce new *questions* and generate new, decisive *models*. We propose then that the project of re-constructing a cognitive mapping amounts to a data-science department within Marxism.

Data and Space

The central question of the "science of history" could be posed as: what determines changes to the form of organized human activity? One can observe different "scales" to this question, each corresponding to a different science and scientific object. However, it is difficult to determine the relative ordering of these scales. Perhaps human activity depends on psychology. Perhaps it depends on international and domestic policies, conflicts, etc. that eventually manifest in individual lives. Or perhaps it is driven by cultural productions and ideological apparatuses. The scale we choose determines our approach to the question, and when we translate data from one scale to another we lose causal information. For example, individual psychology can appear as either the driving force or the effect of other forces depending on our starting point.

If we take this indeterminism-between-scales as a general rule, we find ourselves unable to adhere to the simplistic "base-superstructure" model of society[10]. Instead, we should reframe the question in terms of

8 In terms of linear algebra, we need to select certain *basis* vectors to form a linear space.

9 We can locate the excitement around Piketty 2014 as evidence of the position that data (as well as a proper question) makes a material difference.

10 A model which Kojin Karatani classifies as part of the architectonic impulse in Western thought. See Karatani, 1995

space and transformations between different spaces. In this view, the points of each space are its data, and the difference between points is only visible within that space (or spaces which are equivalent to it). When we transform one space to another, we lose information (differences between points). Therefore, when we attempt to interpret data, we must always ask ourselves which space we are working in, and the tradeoffs we incur when moving between spaces. Every (non-trivial) dataset has an infinite number of possible spaces it can be projected into. For a subject matter as manifold as history, there is no primary space, only those spaces appropriate for modeling causality for a given data set.

Therefore, this text will not directly explore techniques for studying data, but rather attempt to augment our intuition of space[11]. This will allow us to later develop a notion of the "shape", or **topological** properties, of data[12]. Perhaps this will enable a view of the determinations of the value-form in a new, non-reductive manner. To begin, **Euclidean** space in two dimensions can be characterized by the famous formula:

$$a^2 + b^2 = c^2$$

which is the relationship of the sides of a right-angled triangle to its hypotenuse, a fact known as the Pythagorean theorem. If we are given an origin point and a pair of real-number axes[13], called x and y, we get the standard Cartesian plane[14].

11 I have highlighted certain terms to make it easy for the reader to shore up any gaps in technical knowledge.

12 The "shape" of data is the object under scrutiny in the nascent field of Topological Data Analysis (TDA). TDA builds on the notion of "persistent homology" to extract information from data in a way that is resistant to noise. The present text does not aim to provide sufficient background for TDA. However, a good starting point is Curry 2017.

13 When both *a, b* = 1, c will be an irrational number - numbers which cannot be represented as fractions - so we must use the real numbers (which include irrationals).

14 Euclidean space has an inherent notion of distance. The Cartesian plane is just the Euclidean plane with a system of coordinates. In the latter case, we can do something extraordinary. If we hold c constant, and allow the coordinates of the sides to vary, we produce the equation of a circle. This gives rise to the notion of "algebraic curves", equations whose solutions (values which make the equation equal zero) describe shapes in space.

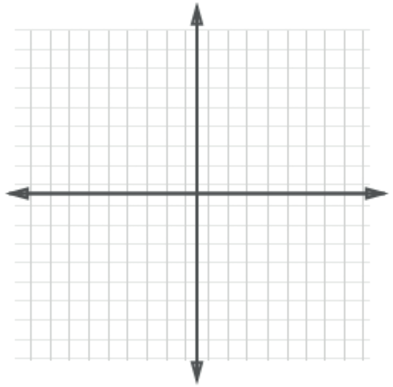

Fig. 1 *Cartesian Plane*

Any point on this plane can be interpreted as the **vertex** of a right angled triangle, where its distance to the origin is the value of the hypotenuse *c*. Likewise the distance between any two points in this plane is the hypotenuse of another right-angled triangle[15]. We can conceive "distance" here to be a very basic type of data, and the Pythagorean formula as a way to transform this data into new information. For example, we can now ask a very common data question: given a set of points and a single point in that set, what are the nearest neighbors of that single point?

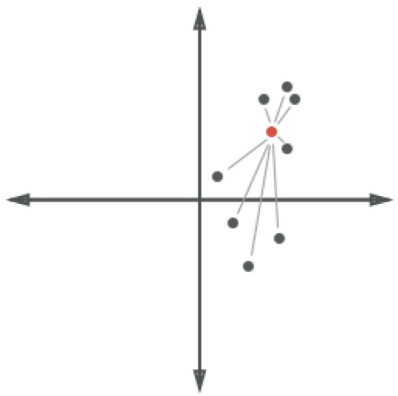

Fig. 2 *The nearest neighbors problem*

The data theoretic perspective then consists of tracking this

15 We recover the Euclidean distance formula proper just by replacing the variables on the left in the Pythagorean theorem with distances between Cartesian components. So $a = x_1 - x_0$, $b = y_1 - y_0$, and so on.

information over time as the distribution of points changes. We call any space in which we can compute distances between points, such as the one just described, a **metric space**. We can produce other metric spaces by changing the formula of Pythagoras to something else, provided that certain conditions hold.[16] We can also vary the dimensions of our space[17].

An important step in enriching the idea of space is to study **transformations** of one space to another. A practical example can be found in Galileo's studies of motion. For Galileo (and Newton), uniform motion and rest are indistinguishable as frames of reference for expressing physical laws. This principle can be expressed as transformations of one **reference frame** to another, taking distance travelled to be invariant. If particle *A* moves 5 meters away from particle *B* at some **velocity**, it is physically equivalent to say that particle *B* has moved 5 meters away from particle *A* at a symmetric velocity[18]. Another more geometric description is to say that the distance between *A* and *B* form the hypotenuse of a triangle which has the same lengths and angles in any reference frame at a given moment. One could therefore imagine various frames (at a given moment) as rotations of one another, and Galileo's principle as a type of **symmetry**[19] insofar as it preserves the Pythagorean relation.

16 More formally, a function is a metric if it is symmetric, positive-valued, and satisfies the triangle inequality. Another example: the metric a + b = c describes a space resembling the rectilinear streets of Manhattan. In three dimensions, it describes a space where points are distributed along a cubic lattice.

17 In the Euclidean context, we can simply add or remove squared terms to the left side of the equation (e.g. $a^2 + b^2 + c^2 = d^2$ in 3 dimensions where d is now the length of the hypotenuse).

18 The velocities are not the same, since we are moving in opposite directions relative to one another. However, the magnitude of the velocities are equivalent.

19 This formalism is called *linear algebra* and rotational symmetries belong to the theory of *groups*.

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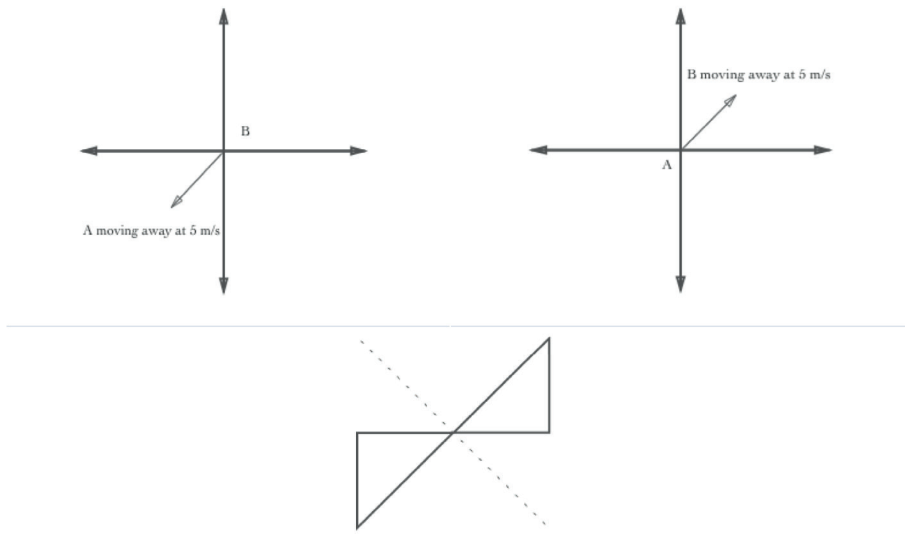

Fig. 3 *Galilean transformation of the coordinates of one space to another, preserving relative distance*

A famous theorem by Noether asserts that the symmetries of a physical system correspond to a quantity which must be conserved in transformations of that system. In the above, we preserved relative distance between different reference frames, but there are symmetries corresponding to energy, momentum, particle "spin", and so on. **Field theory** tracks these quantities at every point in a space and describes ways of transforming them along with the spaces themselves. We can view this as a continuation of the theme of conjoining data with space.

Now, to incorporate time, we can introduce a new parameter t into our Euclidean metric. We can define it as:

$$t^2 + x^2 + y^2 + z^2 = s^2$$

This is the formula for 4-dimensional Euclidean space where time is just another spatial dimension. Yet, Galilean symmetry does not apply to the time dimension as the latter is unidirectional. One can get around the difficulty again by considering a transformation from space to time which would enforce this unidirectional property. In other words, we want to associate to every point in space a time, and to require that transformations of spaces preserve this quantity. Yet, we may also want to view the evolution of a particle system through time, so it is useful to

consider a transformation of moments of time back into a given space. To achieve both, we can start by visualizing "snapshots" of a 3-dimensional space () at each tick of a virtual clock, and arranging these snapshots as successive "slices" along a real number line, called the timeline (). Doing this, we obtain the following picture:

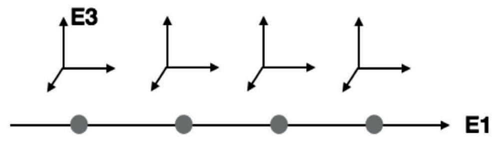

Fig. 4 *The "trivial" line bundle E^1 x E^3*

This construction is an example of a **bundle**, where each snapshot of (Euclidean 3 dimensional space) is a **fiber** "indexed" by a **base space** (the 1 dimensional timeline). Given a particle and some distinguished origin point, we can view a trajectory at each moment in time. By Galilean symmetry, we can also choose an infinite number of reference frames for this trajectory, including one where the particle remains still. However, a reference frame cannot include any information about the particle's future or past state. Time is universal under Galilean transformations because they act on a given fiber but not along the base (time). We can picture uniform motion in this setup as a straight line intersecting with each snapshot at a single point. Non-uniform (accelerating) motion is described by curved lines. Such (straight or curved) lines can be pictured as "embeddings" of the timeline itself in the bundle space. In this view, we are "lifting" the timeline into a larger space, giving it more degrees of freedom. Since there could be many such liftings, each one is aptly called a **cross-section**, to indicate that they cross each fiber at a single point for every point in the base[20].

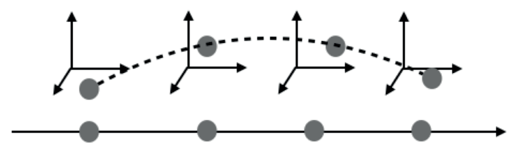

Fig. 5 *Cross-section of the bundle corresponding to non-uniform motion*

20 Another example of a bundle is given by the act of watching a TV show. We are able to rewind, fast forward and jump to various points in the show using the timeline controls on our computer, or a remote control. This implies that our fibers are 2-dimensional pictures on the screen indexed again by a 1-dimensional space. The movement of an object across the screen corresponds to a cross-section. For a gentle introduction to this, see Lawvere, Schanuel 2009, pp. 91-98. For a physics-motivated introduction, see Penrose, 2004, pp. 325-356.

Again, to take a data-theoretic view, we can associate to every point in the base (the timeline) its possible cross sections, without necessarily representing the fibers themselves. Such a perspective may illuminate why this construction is called a bundle:

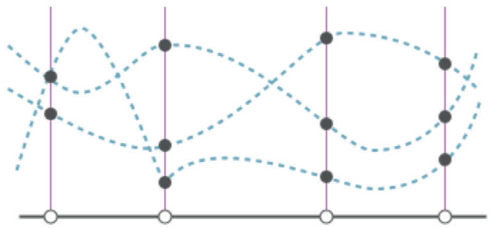

Fig. 6 *A line bundle with "stalks" growing from the bottom, "germs" intersecting with cross-sections*

A question we can now ask is, for a given subset of our base space, what are the cross-sections which preserve certain properties, such as continuity? In the physics of motion prior to Einstein, the **set** of cross-sections was conceptually larger, since there was no known universal bound on the velocity of objects. Therefore, a particle in space at one moment in time could be anywhere else in the next moment, provided it had enough speed. This changes with Einstein, specifically with the advent of the Minkowski metric derived from relativity theory, which can be written as:

$$-c^2t^2 + x^2 + y^2 + z^2 = s^2$$

where *x, y, z* are the usual Euclidean distances, *c* the speed of light, *t* the elapse of time, and *s* the spacetime interval between two events. Notice that the interpretation of points has now changed from being purely spatial to one involving a particular constraint on time. For example, setting *c = 1* and *x, y, z = 0* we obtain the equation for a cone[21] along the time axis (now modeled as a complex axis): *s = t * sqrt(-1)*, and conversely, setting *t = 0* we obtain the Euclidean metric again. This cone is a restriction on not only possible movements of a particle at the origin, but also of any information whatsoever.

One could imagine the values of *x, y, z* and *t* being recorded by satellites orbiting Earth. Special relativity suggests that the relative motion between satellites will cause their respective clocks to drift

21 Technically, a hypercone, since it is expressed in four dimensions.

apart. In order to account for this difference, one exploits symmetries in Minkowski spacetime, just as we did in Euclidean space. This amounts to applying transformations which preserve the spacetime interval *s*, the so-called Lorentz transformations.

Formulating the above in terms of bundles, there is a restriction in the possible cross sections of the bundle due to the fact that nothing can travel faster than light, and that light-speed is constant in all reference frames[22]. This restriction can be expressed by the way in which fibers must be "held together" when we generalize our base space to **manifolds** instead of simple Euclidean spaces. Einstein's general theory of relativity formulates spacetime as inherently curved by energy. The notion of straight lines is replaced by more general **geodesics**, paths which describe this curvature over some set of connected fibers[23].

Traditionally, the problem of connecting fibers arises in the problem of **parallel transport**, that of moving a vector along a surface such that it remains parallel at all times.

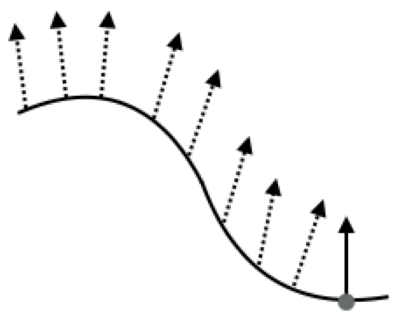

Fig. 7 *Parallel transport of a vector along a curve*

We can augment our intuition by asking ourselves why such a problem is important[24]. If we imagine placing ourselves within a single Euclidean fiber, we should notice that there is no question of the definition of parallel. It is only when tracking changes of points within a

22 Just as Galilean transformations preserve the Pythagorean theorem in all reference frames, the Lorentz transformations preserve the form $x^2 + y^2 + z^2 - t^2$ for any (x, y, z, t).

23 Among other things, this explains why massless particles are influenced by gravity (which Newton's theory could not explain).

24 In physics, this problem leads to the central notion of "gauge" invariance, which concerns preserving a different quantity (field strength) under different particle configurations. An early example of this is the relation between electric and magnetic fields formulated by Maxwell.

succession of fibers (along some subset of the base) that we can observe geodesic motion. A relativistic framework that captures this kind of motion should ideally be able to transform back to the local Euclidean case, when the curvature of space by gravity is negligible. Yet it should also express global features. Earlier we introduced the notion of spaces and transformations between spaces - now we must consider how to characterize "higher-order" features of a system of spaces. For example, how can we distinguish between the following bundles?

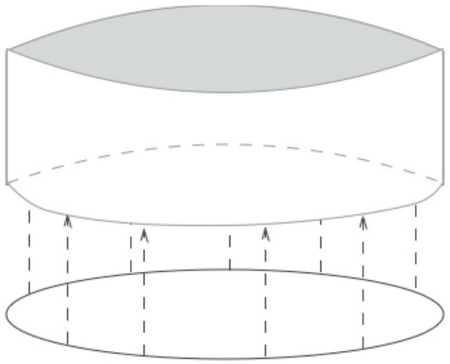

Fig. 8a *A trivial circle bundle* $S^1 \times E^2$

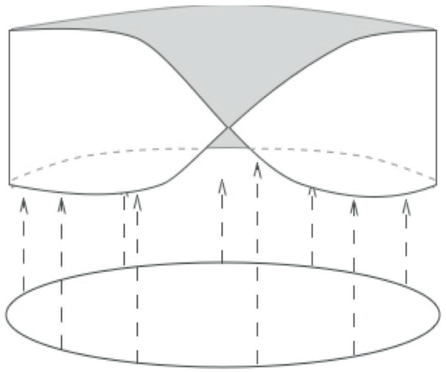

Fig. 8b *A non-trivial bundle over* S^1

Figure 8b depicts fibers that are arranged as slices of a **Mobius band**. This satisfies our definition of a bundle since every point in S^1 (a circle) below has a corresponding E^2 fiber above. Yet when we consider the parallel transport problem, there is an obvious issue: any vector in the bundle moved along the circle will change directions by the time it has reached its starting point again. This feature is invisible at the level of an individual fiber. Moreover, it may be invisible even if we examine most of the fibers together. Only if we take a global view of this space can we verify this property.

Ideology and restrictions to the local case

The concepts described above come from mathematics and physics. Why bring them up in a text which opens on questions of Marxism and political organization? Our position is that the crisis of cognitive mapping discussed in the introduction pertains to a poverty in our intuition of space. When we consider our economic system as a global phenomena, it is tempting to think of the base space as the surface of our planet.

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By assuming this, we already cast problems in terms of geography and physical distance. Yet, it is clear that today, a migrant worker in one country has more in common with one across the world than he does with his neighbors. Those of us with internet access live in a different world than those without. The link between exploitation (and more generally, immiseration) and surplus value seems non-existent when markets can emerge and disappear in an instant, registering only for a moment on computer screens. At the same time, economists and politicians use the same geo-political rhetoric as before, attempting to map incompatible phenomena into the common space of the visible. Here, ideology is a matter of producing false, or reductive, *localizations* of more general phenomena.

To address this, we are attempting to develop an intuition which will aid us in absorbing and modeling data in new, meaningful ways[25]. However, this requires both an education in new formalisms and the critical step of questioning our assumptions. Our current systems of representation are highly susceptible to reducing phenomena to an individual level. This is why modern economics, for example, is built up from several tenuous assumptions about human beings and their self-interest. These assumptions amount to an unspoken metaphysics[26], where price signals are supposedly reflections of aggregated *individual utility*. This serves an ideological purpose: it justifies market activity as the "will of the people" and therefore sacred. Following this line of thinking to the end, if we attempt to constrain the market, we distort its inherently democratic power (where our money counts as our vote), ultimately curtailing individual freedom.

Yet, another viewpoint is that prices are outcomes of a game of specular reasoning and are not determined by utility at all. Under this (Marxist) critical stance, the drive for profit is what sustains the system of prices, including the price of labor which, as labor becomes commoditized, makes a mockery of individual freedom. In both there is a tendency to reduce economic complexity to an issue of individual psychology. For the former, it is a matter of rational self-interest, and in the latter, of class consciousness. However, what our discussion on space entails is that the setting proper for studying this complexity is not individual, but formal.

25 Such a project would obviously require far more than this brief expository text. One would need to gather, at the very least, a working knowledge of topology, statistics, economics and computer science. But we intend to demonstrate that these various fields *could* be combined in novel and interesting ways to aid in political action.

26 This is brilliantly argued in Dupuy 2014

Any political project must construct tools adequate to the phenomena it is attempting to change[27]. In the case of the market, it is clear that the tools must be transnational and at least partially computational. The 2008 crisis was worldwide, and aside from growing distrust of financial capitalists, it has not yielded a proper mechanism for preventing future events. From a game theoretic perspective, if only a subset of nations regulate the market, then remaining nations have more incentive to deregulate. This situation changes only when global externalities are counted in the price of deregulation for an individual agent. A way of viewing these externalities in terms of *intrinsic properties* of a space is a tool the Left must construct.

This is also why it is useful to study modern mathematics, which can be considered a science of equivalences. The term mapping, for example, can be generalized by considering the importance of maps, or **morphisms**, in category theory[28]. In the categoric perspective, all mathematical properties can be characterized by morphisms which transform mathematical objects to other mathematical objects. In most cases, information is lost in the process, and the transformation is one-way. In key cases, however, morphisms are **invertible**. We can find examples in plain functions of numbers, sets, or spaces, but also in proofs which transform one mathematical statement to another. This means that mathematical activity itself produces transformations (of the existing body of statements to new statements) in a sense fully compatible with its own formalism. The basic operation is **composition**, which is the act of producing new morphisms from old ones. Two mathematical objects which are not equivalent (or rather, **isomorphic**) can nevertheless share information via chains of composed morphisms between them.

The scientific procedure generally consists of going in the reverse, which is to say, decomposing morphisms (e.g. physical phenomena) into their factors. This is what we're going to investigate now in the context of space and data. Namely, given a set of points, can we detect their underlying shape? Also, can our methods work regardless of which metric we choose for our space?

27 This principle is formulated thoroughly in Tupinamba, 2014.

28 The rules of category theory of simple, but the game itself is enormous in scope. To form a category one needs **objects** and **morphisms**, where morphisms are defined as having objects as their domain and codomain (source and destination). There is only one operation required to start: composition of morphisms, which produces a new morphism from two or more other ones. This operation is subjected to the rule of associativity (combining morphisms f, g, h is the same as combining f and g first, then h, which is the same as combining g and h first, then f). Finally, every object has at least its identity morphism, which is a morphism going from the object back to it. The game itself consists of finding which categories are equivalent.

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Sheaves

We have seen how the structure of a bundle can be described by taking cross-sections of it. These cross-sections can describe global features not necessarily visible in local regions of the bundle space. We might then become interested in obtaining the cross sections required to reconstruct certain features of their underlying space. Or we might be interested in spaces which may look the same locally but have different global structures. This begins to sound like problems of data-science and computational learning where one wants to approximate a certain *structure* using data and then use that structure to make predictions. Along these lines, we propose to extend the metaphor of cognitive mapping with another one, that of the mathematical **sheaf**[29][30].

Intuitively, a sheaf is a consistent assignment of data to space. Everything lies in this notion of consistency - it tells us why certain assignments will work and others will not. First, we need to generalize the definition of space from the metric and bundle description above[31]. Let us define a topological space as **open**, that is, as a collection of points which do not contain their collective boundary. If we choose any point in an open region, we are able to form a ball, centered on this point, which is fully contained in the space. Alternately, we can say that any point in an open region can be moved by some arbitrary distance and still remain within that region[32]. This property allows us to forego an explicit metric for distances. Let us call **continuous** any function (assigning points in one space to another) which maintains this quality. That is, given an assignment of point A to point B, if B belongs to an open region then the same holds for A. A stronger condition is to require that any point A can be recovered from B, and vice versa. Any such continuous function is called a **homeomorphism**. For example, we can assign

29 Sheaves were invented by the French mathematician Jean Leray while he was interned at a POW camp during the second World War. They were subsequently used by Alexander Grothendieck to axiomatize "homological algebra", a branch of algebraic topology. In an important paper from 1957, Grothendieck establishes that the category of sheaves of abelian groups is the "appropriate setting" for algebraic topology. The notion of a "topos", also essential to modern mathematics, is a generalization of this work. For a good history, see Mclarty 2003.

30 The author is by no means an expert on this topic. However, the aim is to evoke interest in the sheaf and its related notions. That being said, the main reference is Mac Lane and Moerdijk 1992.

31 The "ladder of abstraction" for the notion of space is one of the longest in mathematics. In the following examples we use a basic definition of topological space, that is, a set of points with open subsets obeying certain axioms. Alexandre Grothendieck demonstrated that one could replace these points with objects in a category and still maintain the sheaf conditions.

32 Given this notion of openness, the following axioms must hold for topological spaces:
1. The (finite) intersection of open regions must be open.
2. The union of two or more open regions must be open.
3. The entire space is open and so is its empty space.

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points of a (hollow) circle to a closed disk, but when we attempt to do the reverse, points close together inside the disk will be "torn" from one another. On the other hand, assigning points of a circle to a square is invertible - therefore these spaces are **homeomorphic**. Topology deals with classifying spaces in terms of properties which are independent of metrics. One such property is that of **homology** which, roughly speaking, measures how many n-dimensional holes are in a space. A closely related property is that of **homotopy**, which classifies spaces as equivalent if they can be continuously deformed from one to the other[33].

Immanuel Kant defined "synthesis" as the process of unifying multiple, disparate representations under a single concept[34]. Computational data science aims at a similar goal. In the field of machine vision, for example, the primary task is to train a computer to recognize objects from images. There can be an infinite number of representations of the same object, so this task can be quite formidable. "Deep learning" is a technique for extracting multiple levels, or layers, of a given representation. Concepts such as shape, color, texture, etc. can be derived dynamically insofar as each contribute to the goal of classification. By computing a score for a particular image (with respect to these features) such a deep learning network can determine that an image is a representation of a particular object. This requires "training" the network on test representations, giving it a sense of the factors needed to transform an image into its classification. Clearly, the space of representations is open in the sense defined above, since the computer must be able to correctly classify images it has never seen before. In other words, the deep learning network constructs intermediate spaces from representations and the classification space (a yes or no in many cases). At the heart of such techniques is the spatial abstraction of data.

If we add arrows between open regions of a space whenever one region is contained in another, terminating with the entire space and starting with the empty region, we get something akin to the following[35]:

33 A good introduction available online is Hatcher 2010.

34 Here is an interesting intersection with modern artificial intelligence, since for Kant there is no synthesis which is not immediately conscious. For example: "This thoroughgoing identity of the apperception of a manifold which is given in intuition contains a synthesis of representations, and is possible only through the consciousness of this synthesis." from Kant 1998, p. 247

35 A lattice is formally defined as a partially ordered set where any two elements have a *join* and a *meet*, that is, a common "parent" and "descendent" respectively.

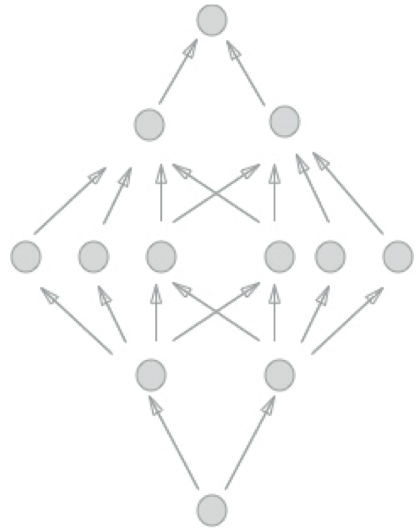

Fig. 9 *A topology depicted as a lattice*

If we simply replace "contains" with "is greater than", then this is an order relation. However, sometimes two regions are neither greater than nor less than each other (i.e. are siblings). This is therefore a **partial ordering**. This ordering is another type of data similar to distance. Just as we can classify transformations of points of a space by how much information (differences between points) is lost (cannot be recovered by an inverse transformation), we can split transformations which preserve open regions and their relative ordering from those that don't. In fact, given an adequately rigorous idea of "open regions", one doesn't need to refer to points at all.

A **presheaf** assigns open regions of a space to data such that subsets of data correspond to subregions. Going from a region to a sub-region produces a **restriction** on the associated data. One can imagine two "screens", one containing a space and another some information. One is allowed two actions: to "select" an area of the space and to "shrink" the current selection. When we select different regions, we see a corresponding change of information on the other screen. When we shrink our selection, we see the information shrink respectively.

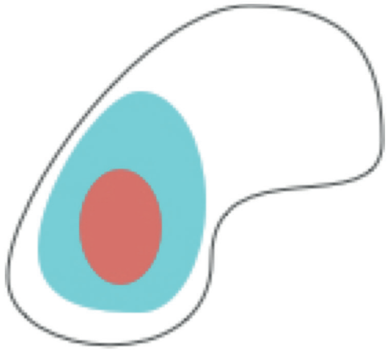

Fig. 10a *A region and its sub-regions*

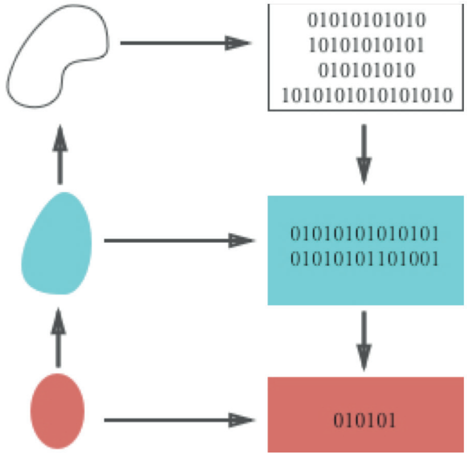

Fig. 10b *Inclusion of a space yields a restriction map (contravariance)*

However, there is a slight problem. Sometimes, selecting two different regions produce the same data. Also, there is no way to combine two selections together in a determined way. This is what a sheaf provides. To move from a presheaf to a sheaf, we need to add the following two constraints:

1. Uniqueness - If two regions have the same data associated to them, they are the same region.
2. Gluing - Two regions can be glued together if the data associated to their intersection agree.

Taking these constraints together, a sheaf determines a unique, global **assignment** of data for a given space. This is especially useful in contexts where the space in question is not given beforehand, but must be assembled or approximated using computational methods. In intuitive terms again, the second condition gives us a third action: "gluing" two selections together forms a (unique) third selection. One can imagine this roughly as a puzzle where we do not yet have all the pieces, but if we guess an adequate space for the pieces we do have, we can be assured that these pieces belong to a unique construction (i.e. there are no extraneous or duplicate pieces).

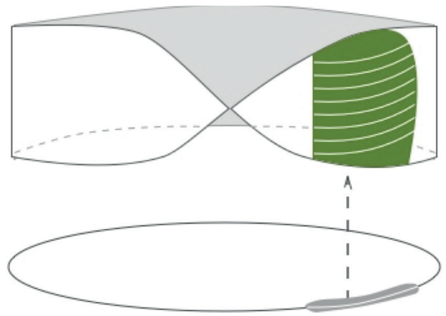

Fig. 8a *The presheaf of sections on a non-trivial bundle - the green region is assigned the white lines corresponding to cross-sections over S^1*

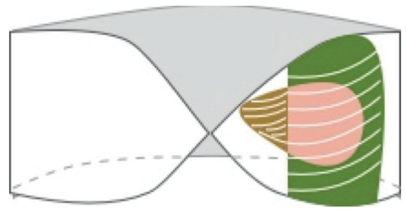

Fig. 8b *We can glue the green and brown regions as long as they agree on their overlap*

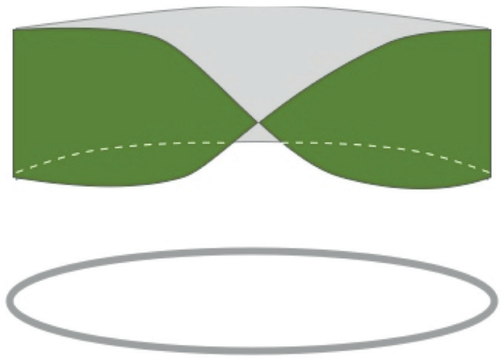

Fig. 8c *Extending the region to the entire circle, non-vanishing cross-sections disappear due to the non-orientable nature of the space.*

The sheaf therefore highlights the ways in which topological features may determine data. We call the examples in Fig. 8 a "sheaf of sections of a bundle", but sheaves may have any type of value. In the case of machine vision, the "base" is the image itself, the "bundle" the feature space of an image (e.g. its "redness" or its "circle-ness" arranged as linear bases), and the sheaf the assignment of values to the feature space. We can then formulate the question: what are the invariants of a space which limit its consistent assignments? Just as information in the

Einstein-Minkowski universe is bounded by the speed of light, the region of non-vanishing cross sections of a vector bundle can be "bounded" by its twist. These bounds are the invariants of the space which are generally unknown beforehand. From this (rather cursory) look at sheaves and the mathematics of assigning data to space, we can now attempt a broader claim regarding "cognitive mapping".

Price, Value, and Space

We tend to default to a *personal framework* when faced with world events and economic data. In this framework, events are caused by individuals consciously acting to achieve their goals. This inevitably paints phenomena in humanist and moral colors. With the rise of truly complex systems which govern our lives, perhaps it is time to also consider formal, a-cognitive methods. In the following, we offer nascent ideas for incorporating topology and computation into a map of the economy.

At the outset, it is important to denote the difference between price and value: whereas market prices are determined by exchange activity (i.e. supply and demand), value is determined by a hidden variable in the system. In Marx's theory, it is "abstract labor time", a socially determinate measure of labor needed to produce a given commodity. Whether there exists a formula to compute price from value is still unresolved. Yet, we know that such formula, if it were invertible, would trivialize Marx's project since it would mean that price and value represent the same thing. In our view, the discussion of the "transformation problem" can be made fruitful if we consider value as a space with topological properties. This would mean that a single variable (labor-time) is inadequate to capture this space. Marx's own solution to this problem involved introducing a variable representing the "organic composition" of capital[36]. Without entering the debate surrounding the coherence or validity of this approach, we can observe that it amounts to enlarging the space of value with additional variables that could explain the dynamism of prices (without thereby reducing value to price). However, as we have just introduced, we can study space independent of any chosen metrics. Let us consider then that value is comprised of open regions which can become "flattened" into prices. In other words, we assume that the space of value is inherently larger than that of prices.

36 See again Heinrich 2013 and the responses by Carchedi and Roberts 2013. In general, the debate surrounds the question of whether "the tendency of the rate of profit to fall" is a necessary component of Marx's theory.

Accordingly, we can assign (or lift) prices into this larger space. Intuitively, commodities with vastly different properties can nevertheless have the same price. Yet, how does this assignment occur? When market activity takes place, prices take on logical constraints which are immanent to a given economic situation. This is usually modeled as a game where multiple players attempt to maximize their gain. Each game, provided certain conditions hold, yields one or more possible equilibrium points - those in which each player has found a strategy which cannot be improved upon. In the game of the market, the set of possible strategies consists of when to buy or sell commodities and at what prices. There can be markets within markets (which are not necessarily local to any region of the Earth), thus they can exhibit a nested structure similar to that of the open sets of a space. This process is supposed to converge to equilibrium levels as we go further up the chain. Intuitively, the convergence of prices is explained by the fact that any discrepancy for a given commodity between markets allows a player to profit (by buying low and selling high, for example). In this way, the market is a game which also behaves like a space. It follows that we have the ingredients to make a pre-sheaf, namely, one which assigns possible prices (corresponding to strategies).

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The notion of an "efficient market" may be recast in terms of a sheaf. The uniqueness property tells us that every strategy has a determined payoff. The gluing property tells us that the various markets *agree* on the price of mutual commodities. These conditions then correspond to the thesis of converging prices and strategies. In the case where data does not fit the consistency criteria (gluing and uniqueness), mathematicians may use an algorithm called **sheafification** which modifies the assigned data to fit a sheaf. Accordingly, we can conceive the activity of the market as a machine for such sheafification of prices.

A price serves as a signal that a commodity may be over or under priced relative to some fixed imaginary price. Whoever "fixes" this inefficiency first makes a profit. Ideally, this leads to prices which are consistent across the global market at any given moment. We could call this the *Hayekian*[37] picture of how prices come to be and the function they serve. For Hayek, the market resembles an omni-intelligent force because it incorporates unsystematic, time-sensitive knowledge in its system of prices. By reacting immediately in a decentralized manner, the market can resolve coordination problems between various actors even in cases of total anonymity[38]. Yet, between the individuals "on the spot" and the

37 Hayek 1945

38 This is also the allure of cryptocurrencies, which promise to "free" money from its institutional shackles. The ideological underpinning of such movements can still be found in Hayek.

global convergence of prices, there is a whole series of intermediate unknowns, including firms and institutions, asymmetric information and power structures, etc. Furthermore, even though the market *can* extract useful signals from knowledge, it also inputs these signals back in - leading to the possibility of a nonlinear, non-equilibrium system. Due to this complexity, the reliability of the market to become efficient is taken as a given. This leads us back to the question of cognitive mapping. If we assume that the space of value is well-understood (where, for example, one can distinguish independent random variables), we can join Hayek in celebrating the miracle of price system. However, if this space is non-trivial, then we cannot trust that a sheaf of prices exists. This seems to be the case when we consider the role of credit in sustaining the system and the culpability of complex financial instruments in recent crises. Instead of thinking of the market as always in the process of converging to equilibrium, we should think of it as attempting to stave off crisis by producing its own formal means of consistency. By identifying the market as a continual process of sheafification, we may be able to computationally map this process and therefore find critical points of intervention. To do this, we have to shed our assumptions about convergence of prices and instead incorporate data generated by global crises.

What Hayek's approach misses is how the price system restructures the very knowledge that sustains it. This restructuring is generally taken as a form of progress - as technology improves, workers are freed to specialize, which gives rise to the "knowledge-class". This in turn leads to increased productivity as business firms transform under a confluence of different fields. However, knowledge is a form which inherently resists commodification. Attempts to create boundaries around it in order to make it rentable are transient, as it has (near-)zero reproduction cost. Businesses quickly adopt the latest technologies and automation techniques, and the outcome is that less workers are needed. The correlate to the knowledge class is therefore the transiently or permanently unemployed class.

In assigning prices to the space of value, human society achieves dynamic growth and coordination, but this process then transforms value itself. Along these lines, what if the value space has topological properties which prevent a *consistent global* assignment of prices? This is not simply asserting that conditions are never ideal due to external factors. It is asserting rather that the sheafifying process inherently fails because of factors which are not visible in local assignments (which may appear efficient after all). These topological factors only appear as **singularities**, or points where the sheaf of prices break down. In other words, they would be "topological generators" of crises.

We should avoid the trap of moralizing the problems we face today. It is not greed, nor even negligence, which lead to crisis, but features of the system itself. Students of Marx should acknowledge capitalism as a complex machine, not as conspiracy or manifestation of evil. Non-Marxists should acknowledge the non-equilibrium nature of the market. And we should consider political-economic decisions as those which force a price-assignment that considers the entire space, rather than the local, profit-maximizing ones. This may include increasing benefits, education, etc. insofar as they are counted as part of the price of a laborer. Yet, we should view these decisions outside of the welfare-state context, that is, not simply as preserving a standard of life, but as producing a new space of value. The true metric for change is not simply one of "economic equality", which can be a red herring for real transformation, but forcing changes in what is invariant in the existing space. Answering to this would amount to a real map of politics into the economy.

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